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Butterworth Low pass filter :

→ It is necessary to first develop a relationship between analog transfer function $H_a(S)$ and square of its magnitude response $|H_a(j\Omega)|^2$.

→ The freq response of an analog filter is given by

$$H_a(j\Omega) = H_a(S)\big|_{s=j\Omega}$$

$$|H_a(j\Omega)|^2 = H_a(j\Omega)\, H_a^*(j\Omega)$$

$$= H_a(j\Omega)\, H_a(-j\Omega)$$

$$= H_a(S)\, H_a(-S)\big|_{s=j\Omega}$$

Or $$H_a(S)\, H_a(-S) = |H_a(j\Omega)|^2\big|_{\Omega=\frac{s}{j}}$$

→ The poles and zeros occur in complex pairs. If $H_a(S)$ is to represent a causal and stable filter, then all its poles must lie within the left half of s-plane.

→ Find the poles of $H_a(S) \cdot H_a(-S)$ and select the poles in left half for $H_a(S)$

As $S = j\Omega$

$S^2 = -\Omega^2$

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$$H_a(j\Omega) = \left| \frac{1}{\left(1 + (\Omega/\Omega_c)^{2N}\right)^{0.5}} \right| \quad \left(\frac{s}{j\Omega_c}\right)^{2N}$$

$$H_a(s)\,H_a(-s) = |H_a(j\Omega)|^2 \Big|_{\Omega = \frac{s}{j}} = \frac{1}{1 + \left(\frac{-s^2}{\Omega_c^2}\right)^N}$$

The poles of $H_a(s)\,H_a(-s)$ are formed by solving the equation

$$1 + \left(\frac{-s^2}{\Omega_c^2}\right)^N = 0$$

$$\Omega_c^{2N} + (-s^2)^N = 0$$

$$(-s^2)^N = -(\Omega_c)^{2N} \Rightarrow (-1)^N s^{2N} = -(\Omega_c)^{2N}$$

Since $e^{j(2k+1)\pi} = -1$ for $k = 0, 1, 2, \ldots 2N-1$

$$(-1)^N s^{2N} = e^{j(2k+1)\pi} \cdot (\Omega_c)^{2N} \quad k = 0, 1 \ldots 2N-1 \quad (1)$$

Case I : When N is even

$$(-1)^N = 1$$

$$s^{2N} = e^{j(2k+1)\pi} \cdot (\Omega_c)^{2N}$$

$$s = \Omega_c\, e^{j\pi(2k+1)/2N} = \Omega_c\, e^{j\left(\frac{\pi}{2N} + \frac{k\pi}{N}\right)}$$

when N is even, Poles are

$$P_k = \Omega_c\, e^{j\left(\frac{\pi}{2N} + \frac{k\pi}{N}\right)}, \quad k = 0, 1 \ldots 2N-1$$

Case II : When N is odd,

$$(-1)^N = -1$$

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$$(-1)^N s^{2N} = e^{j(2k+1)\pi} \cdot (\Omega_c)^{2N}$$

$$-s^{2N} = e^{j2k\pi} \cdot e^{j\pi} \cdot (\Omega_c)^{2N}$$

$$= -e^{j2k\pi} \cdot (\Omega_c)^{2N}$$

$$s^{2N} = \Omega_c^{2N} \cdot e^{j2k\pi} \Rightarrow s = \Omega_c e^{j\frac{k\pi}{N}}$$

$$k = 0, 1 \dots 2N-1$$

When N is odd, the poles are

$$P_k = \Omega_c e^{jk\pi/N}$$

$$P_0 = \Omega_c e^{j0}$$

$P_1 = \Omega_c e^{j\pi/N}$ and so on.

→ There are 2N poles of $H_a(s) H_a(-s)$ which are equally distributed on a circle of radius Ω_c with an angular spacing of $\frac{\pi}{N}$ radians.

→ The poles are symmetrically located with respect to $j\Omega$ axis.

→ To ensure causality and stability,

$$H_a(s) = \frac{\Omega_c^N}{\prod\limits_{\text{LH Poles}} (s - P_k)}$$

$(s-P_0)(s-P_1) \dots (s-P_N)$

Q Determine the system function $H_a(S)$ for Butterworth filter. Determine the poles of filter for $N=2$. Sketch the poles of $H_a(S)H_a(-S)$ and determine the Butterworth LPF system function $H_a(S)$

Sol $N=2$ (even)

The poles of $H_a(S)H_a(-S)$ are

$$P_K = \Omega_c e^{j\left(\frac{\pi}{2N} + \frac{K\pi}{N}\right)} \quad K=0,1 \ldots 2N-1$$

$$P_K = \Omega_c e^{j\left(\pi/4 + \frac{K\pi}{2}\right)} \quad K=0,1,2,3$$

$K=0$, $P_0 = \Omega_c e^{j\pi/4} = \Omega_c(0.707 + j0.707)$

$K=1$, $P_1 = \Omega_c e^{j\left(\frac{\pi}{4} + \frac{\pi}{2}\right)} = \Omega_c e^{j3\pi/4}$

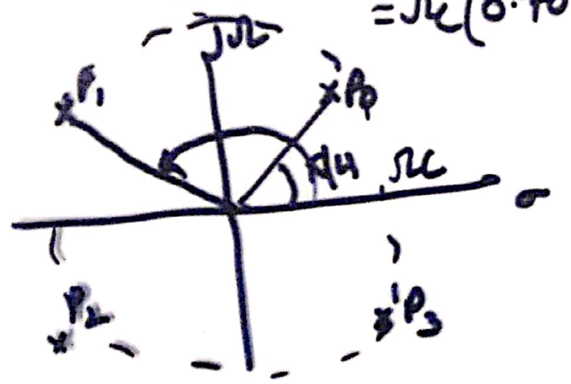
$= \Omega_c(-0.707 + j0.707)$

$K=2$, $P_2 = \Omega_c e^{j(\pi/4 + \pi)} = \Omega_c e^{j5\pi/4}$

$= \Omega_c(-0.707 - j0.707)$

$K=3$, $P_3 = \Omega_c e^{j(\pi/4 + 3\pi/2)} = \Omega_c e^{j7\pi/4}$

$= \Omega_c(0.707 - j0.707)$



$re^{j\theta}$

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To ensure causality and stability,

$$H_a(S) = \frac{\Omega_c^N}{\prod_{LHP}(S-P_k)} = \frac{\Omega_c^2}{(S-P_1)(S-P_2)}$$

$$= \frac{\Omega_c^2}{(S-\Omega_c e^{j3\pi/4})(S-\Omega_c e^{-j3\pi/4})} = \frac{\Omega_c^2}{S^2 - 2\Omega_c S\cos 3\pi/4 + \Omega_c^2}$$

$$H_a(S) = \frac{\Omega_c^2}{S^2 - \sqrt{2}\,\Omega_c S + \Omega_c^2}$$

the normalized system function ($\Omega_c = 1$ rad/s) is $H_a(S) = \dfrac{1}{S^2 + \sqrt{2}S + 1}$

Determination of Filter Parameters (order N and cut off freq. Ω_c) of Butterworth filter:

Case I: When specifications given are δ_1, δ_2, Ω_p & Ω_s

$$|H_a(j\Omega)| = \frac{1}{\left(1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right)^{0.5}}$$

The desired parameters of Butterworth filters are obtained by considering LPF with the specifications

$$\delta_1 \le |H(j\Omega)| \le 1 \qquad 0 \le \Omega \le \Omega_p$$

$$|H(j\Omega)| \le \delta_2 \qquad \Omega_s \le \Omega \le \pi$$

Max. Passband attenuation $\Rightarrow A_p = -20\log_{10}\delta_1$

Min. Stopband attenuation $\Rightarrow A_s = -20\log_{10}\delta_2$

At $\Omega = \Omega_p$, $|H_a(j\Omega_p)|^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} = \delta_1^2$

At $\Omega = \Omega_s$, $|H_a(j\Omega_s)|^2 = \frac{1}{1+(\Omega_s/\Omega_c)^{2N}} = \delta_2^2$

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \frac{1}{\delta_1^2} - 1 \quad \text{— (1)}$$

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = \frac{1}{\delta_2^2} - 1 \quad \text{— (2)}$$

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Dividing ① & ②,

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{\frac{1}{\delta_2^2}-1}{\frac{1}{\delta_1^2}-1} \Rightarrow 2N \log_{10}\frac{\Omega_s}{\Omega_p} = \log_{10}\left(\frac{\frac{1}{\delta_2^2}-1}{\frac{1}{\delta_1^2}-1}\right)$$

$$N = \frac{\frac{1}{2}\log_{10}\left(\left(\frac{1}{\delta_2^2}-1\right)/\left(\frac{1}{\delta_1^2}-1\right)\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

From ①, $\frac{\Omega_p}{\Omega_c} = \left(\frac{1}{\delta_1^2}-1\right)^{1/2N}$

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{\delta_1^2}-1\right)^{1/2N}}$$

Case II : When the specifications given are A_p, A_s, Ω_p & Ω_s.

The magnitude response specifications (in dB) of LPF is given by

$$-A_p \le |H_a(j\Omega)|_{dB} \le 0 \quad , 0 \le \Omega \le \Omega_p$$

$$|H_a(j\Omega)|_{dB} \le -A_s \quad , \Omega_s \le \Omega \le \pi$$

At $\Omega = \Omega_p$

$$-A_p = 20 \log_{10} |H_a(j\Omega_p)|$$

$$-A_p = 10 \log_{10} |H_a(j\Omega_p)|^2$$

$$-A_p = 10 \log_{10}\left(\frac{1}{1+(\Omega_p/\Omega_c)^{2N}}\right)$$

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$$A_p = -10\log_{10}\frac{1}{1+(\Omega_p/\Omega_c)^{2N}} \quad \text{--- (3)}$$

At $\Omega = \Omega_s$,

$$-A_s = 20\log_{10}|H_a(j\Omega_s)|$$

$$-A_s = 10\log_{10}|H_a(j\Omega_s)|^2$$

$$-A_s = 10\log_{10}\frac{1}{1+(\Omega_s/\Omega_c)^{2N}}$$

$$A_s = -10\log_{10}\frac{1}{1+(\Omega_s/\Omega_c)^{2N}} \quad \text{--- (4)}$$

from (3),

$$\frac{A_p}{10} = -\log_{10}\frac{1}{1+(\Omega_p/\Omega_c)^{2N}}$$

$$\frac{A_p}{10} = \log_{10}\left(1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}\right)$$

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{A_p/10} - 1 \quad \text{--- (5)}$$

from (4), $\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = 10^{A_s/10} - 1$

$$\left(\frac{\Omega_p}{\Omega_s}\right)^{2N} = \frac{10^{0.1A_p}-1}{10^{0.1A_s}-1} \Rightarrow 2N\log_{10}\left(\frac{\Omega_p}{\Omega_s}\right) = \log_{10}\left(\frac{10^{0.1A_p}-1}{10^{0.1A_s}-1}\right)$$

$$N = \frac{\frac{1}{2}\left(\log_{10}\left(\frac{10^{0.1A_p}-1}{(10^{0.1A_s}-1)}\right)\right)}{\log_{10}\left(\frac{\Omega_p}{\Omega_s}\right)}$$

from (5), $\Omega_c^{2N} = \frac{\Omega_p^{2N}}{10^{0.1A_p}-1}$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1A_p}-1)^{1/2N}}$$

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Design Procedure:

① Begin with Magnitude response specification of digital filter and transform them into analog filter specifications by using impulse invariant or bilinear transformation method. Calculate bandedge frequencies

$$\Omega_p = \frac{\omega_p}{T} \quad , \quad \Omega_s = \frac{\omega_s}{T}$$

OR

$$\Omega_p = \frac{2}{T} \tan\frac{\omega_p}{2} \quad , \quad \Omega_s = \frac{2}{T} \tan\frac{\omega_s}{2}$$

② Calculate order N of filter

③ Calculate cut off freq. Ω_c of filter

④ Determine the analog filter system function $H_a(s)$

⑤ Transform $H_a(s)$ to $H(z)$.

Ques. Using Bilinear transformation, design a butterworth filter which satisfies the condition

$$0.8 \le |H(e^{j\omega})| \le 1 \quad , \quad 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.2 \quad , \quad 0.6\pi \le \omega \le \pi$$

Assume T = 1 sec.

Ques Using Bilinear transformation, design a Butterworth filter which satisfies the condition

$$0.8 \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.2, \quad 0.6\pi \le \omega \le \pi$$

Sol

$\delta_1 = 0.8 \qquad \omega_p = 0.2\pi$

$\delta_2 = 0.2 \qquad \omega_s = 0.6\pi$

Step I – Determination of Analog filter edge frequencies

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}, \quad \Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$$

$$\Omega_p = \frac{2}{1}\tan 0.1\pi = 0.65 \text{ rad/s}$$

$$\Omega_s = \frac{2}{1}\tan 0.3\pi = 2.75 \text{ rad/s}$$

Step II :

$$N = \frac{\frac{1}{2}\log_{10}\left(\left(\frac{1}{\delta_2^2}-1\right)\Big/\left(\frac{1}{\delta_1^2}-1\right)\right)}{\log_{10}(\Omega_s/\Omega_p)}$$

$$= \frac{\frac{1}{2}\log_{10}\left(\left(\frac{1}{(0.2)^2}-1\right)\Big/\left(\frac{1}{(0.8)^2}-1\right)\right)}{\log_{10}\left(\frac{2.75}{0.65}\right)} = 1.30$$

$N \approx 2$

Step III :

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{\delta_1^2}-1\right)^{1/2N}} = \frac{0.65}{\left(\frac{1}{(0.8)^2}-1\right)^{1/4}} = 0.75 \text{ rad/s}$$

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Step IV : For N=2(even), the poles of $H_a(s)\,H_a(-s)$ are

$$P_K = \Omega_c e^{j\left(\frac{\pi}{2N}+\frac{K\pi}{N}\right)}, \quad K = 0,1 \ldots 2N-1$$

$K = 0,1,2,3$ $\qquad re^{j\theta}$

$$P_0 = \Omega_c e^{j\pi/4} = 0.75\, e^{j(\pi/4)} =$$

$$P_1 = \Omega_c e^{j(\pi/4+\pi/2)} = 0.75\, e^{j3\pi/4}$$

$$P_2 = \Omega_c e^{j(\pi/4+\pi)} = 0.75\, e^{j5\pi/4}$$

$$P_3 = \Omega_c e^{j(\pi/4+3\pi/2)} = 0.75\, e^{j7\pi/4}$$

To ensure stability, the left hand of s-plane poles are identified

$$H(s) = \frac{\Omega_c^N}{\prod_{LHP}(s-P_k)} = \frac{\Omega_c^2}{(s-P_1)(s-P_2)}$$

$$= \frac{\Omega_c^2}{(s-0.75e^{j3\pi/4})(s-0.75e^{j5\pi/4})}$$

$$H(s) = \frac{0.56}{s^2+1.06s+0.56}$$

Using Bilinear Transformation,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}}$$

$$H(z) = \frac{0.56}{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.06\times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.56}$$

$$H(z) = \frac{0.56 + 1.12z^{-1} + 0.56z^{-2}}{6.68 - 6.88z^{-1} + 2.44z^{-2}}$$

$$H(z) = \frac{0.0843 + 0.16z^{-1} + 0.08z^{-2}}{1 - 1.02z^{-1} + 0.36z^{-2}}$$

Find the system function H(z) of digital filter that meets the following specification

(a) 1-dB ripple in passband $0 \le |\omega| \le 0.3\pi$

(b) At least 40 dB attenuation in stop band $0.8\pi \le |\omega| \le \pi$

Use Bilinear transformation method.

Sol.

$$A_p = 1\,dB$$
$$\omega_p = 0.3\pi$$
$$A_s = 40\,dB, \quad \omega_s = 0.8\pi$$

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = \frac{1.0191}{T}$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = \frac{6.1554}{T}$$

$$N = \frac{\frac{1}{2}\log_{10}\left(\frac{10^{0.1A_p}-1}{10^{0.1A_s}-1}\right)}{\log_{10}\left(\frac{\Omega_p}{\Omega_s}\right)} = \frac{\frac{1}{2}\log_{10}\left(\frac{10^{0.1}-1}{10^{4}-1}\right)}{\log_{10}\left(\frac{1.0191}{6.1554}\right)}$$

$$= 2.93 \simeq 3$$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1A_p}-1)^{1/2N}} = \frac{1.0191/T}{(10^{0.1}-1)^{1/6}} = \frac{1.2765}{T}$$

For N=3 (odd), the poles of $H(s)\cdot H(-s)$ are $P_K = \Omega_c e^{jK\pi/N}$ $K = 0, 1 \dots 2N-1$

$= 0, 1, 2, 3, 4, 5$

$$P_0 = \frac{1.2765}{T} e^{j0}, \quad P_3 = \frac{1.2765}{T} e^{j\pi}$$

$$P_1 = \frac{1.2765}{T} e^{j\frac{\pi}{3}}, \quad P_4 = \frac{1.2765}{T} e^{j\frac{4\pi}{3}}$$

$$P_2 = \frac{1.2765}{T} e^{j2\pi/3}, \quad P_5 = \frac{1.2765}{T} e^{j\frac{5\pi}{3}}$$

$$H(s) = \frac{\Omega_c^N}{(s-P_2)(s-P_3)(s-P_4)} = \frac{\left(\frac{1.2765}{T}\right)^3}{\left(s - \frac{1.2765}{T} e^{j2\pi/3}\right)\left(s - \frac{1.2765}{T} e^{j\frac{4\pi}{3}}\right)\left(s - \frac{1.2765}{T} e^{j\pi}\right)}$$

$$H(s) = \frac{\left(\frac{1.2765}{T}\right)^3}{s^3 + \frac{2.553}{T} s^2 + \frac{3.259}{T^2} s + \frac{2.0801}{T^3}}$$

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{z-1}{z+1}} = \frac{0.0776 + 0.2328z^{-1} + 0.2328z^{-2} + 0.0776z^{-3}}{1 - 0.8002z^{-1} + 0.5039z^{-2} - 0.083z^{-3}}$$